

Analysis on flag manifolds and Sobolev inequalities

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Abstract

To Joseph A. Wolf, with admiration

Analysis on flag manifolds G/P has connections to both representation theory and geometry; in this paper we show how one may derive some new Sobolev inequalities on spheres by combining rearrangement inequalities with analysis of principal series representations of rank-one semisimple Lie groups. In particular the Sobolev inequalities obtained involve hypoelliptic differential operators as opposed to elliptic ones in the usual case. One may hope that these ideas might in some form be extended to other parabolic geometries as well.¹

Introduction

J. A. Wolf has worked in and has made lasting contributions to large areas of mathematics, including Riemannian geometry, complex geometry, representations of Lie groups, infinite-dimensional Lie groups, and the role of flag manifolds from many points of view. Of particular importance is his study of boundary components of Riemannian Hermitian symmetric spaces where the role of parabolic subgroups P in semisimple Lie groups G is elucidated. At the same time he has treated many aspects of induced representations $\text{Ind}_P^G(W)$ and the relation between the geometry of the flag manifold $S = G/P$ and the analysis of representations in vector bundles over S .

In this paper we shall consider such parabolically induced representations $\pi_\lambda = \text{Ind}_P^G(L_\lambda)$ where λ is a parameter for a line bundle over S . The aim is to understand the relation between the detailed structure of π_λ , in particular the restriction to a maximal compact subgroup $K \subset G$, the eigenvalues of some standard intertwining operators, and certain Sobolev inequalities in the space of sections of the corresponding line bundle involving natural

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differential operators. These operators will reflect the geometry of S , in particular understood as a *parabolic geometry*, such as for example conformal geometry or the usual CR-geometry; in addition, there will be a quaternionic analogue of the usual CR-geometry as well as an octonionic analogue. In particular the operators that arise are hypoelliptic differential operators as opposed to elliptic ones in the usual case.

Some of our results will be known to experts, e.g. in parabolic geometry with regards to covariant differential operators, and in representation theory with connections to the structure of principal and complementary series representations, but probably their combination is new, in particular the ensuing Sobolev inequalities. Prominent examples of the differential operators in question will be the Yamabe operator appearing in conformal differential geometry, and also the CR-Yamabe operator from classical CR-geometry [10].

The main result is Theorem 2.1, which gives a bound on the entropy of a function on a sphere, viewed as a flag manifold for a rank-one simple Lie group G , in terms of the smoothness of the function; the point is here that the smoothness is only measured in certain directions in each tangent space, corresponding to a natural distribution. This distribution is the structure that is directly related to the structure of G , and it provides the relevant parabolic geometry of the sphere in question.

Also, we have included in the final section a new proof of the logarithmic Sobolev inequality by L. Gross [6] for the Gauss measure; this proof has the advantage of potentially extending to a similar inequality on the Heisenberg group, following as a corollary to our main Theorem 2.1.

Dedication. It is a pleasure to let this paper be part of a tribute to Joseph A. Wolf for his mathematical work and continued energy in revealing new insights, for his contributions as teacher and colleague to differential and complex geometry, Lie groups, representations, and knowledge in general.

1 Geometry of the rank-one principal series

Let G be a noncompact connected semisimple Lie group with finite center; later we shall assume that G is of split rank-one. The Lie algebra of G is denoted by \mathfrak{g} . $K = G^\theta \subset G$ is a maximal compact subgroup corresponding to the Cartan involution θ , and we use the same letter for the differential giving rise to the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$$

into ± 1 eigenspaces respectively. We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and for $\alpha \in \mathfrak{a}^*$ let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (\forall H \in \mathfrak{a})[H, X] = \alpha(H)X\},$$

so we have the set of roots $\Delta = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}$. We choose a positive system $\Delta^+ \subset \Delta$. As usual [14] [15] we have the spaces $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{g}_0$ and $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ as well as the corresponding analytic subgroups $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$, and the minimal parabolic subgroup $P = MAN$, where $M = Z_K(\mathfrak{a})$, the centralizer of \mathfrak{a} in K , has Lie algebra \mathfrak{m} .

We shall be interested in representations induced from characters of P , (*scalar principal series representations* of G) namely with $2\rho = \sum_{\alpha \in \Delta^+} m_\alpha \alpha$, $m_\alpha = \dim \mathfrak{g}_\alpha$, and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we consider $V_\lambda = \text{Ind}_P^G(\chi_\lambda)$, the representation space of sections of the line bundle over S corresponding to the character

$$\chi_\lambda(man) = a^{\rho+\lambda}.$$

The action of $g \in G$ is by left translation and denoted by $\pi_\lambda(g)$ and we sometimes also use the name π_λ for the representation. We identify $S = G/P = K/M$ and realize our induced representation in $L^2(S)$ (normalized K - invariant measure) as

$$\pi_\lambda(g)f(\xi) = a(g^{-1}\xi)^{-\lambda-\rho}f(g^{-1} \cdot \xi),$$

where $\xi = kM \in S$, $g \cdot \xi$ denotes the G -action on S , and $a(g\xi)$ denotes the A - component in the $KMAN$ decomposition of gk . For $\lambda \in i\mathfrak{a}^*$, π_λ is unitary; and the smooth vectors are just the smooth functions on S .

Our aim is to combine some of the results in [3] and [12] with estimates by E. Lieb [13] and W. Beckner [1], [2] in order to obtain some new Sobolev type inequalities; they rely on some classical rearrangement inequalities (see the Appendix), and may be thought of as new instances of logarithmic Sobolev inequalities as found and studied in particular by L. Gross [6], [7]. While we expect many of the results to hold in greater generality, we shall from now on consider the rank-one case, i.e. assume A is one-dimensional, so that in particular the parabolic subgroup P is also a maximal parabolic subgroup.

This means that we are dealing with (up to coverings) four cases:

- the real case $G = SO_o(1, n+1)$,
- the complex case $G = SU(1, n+1)$,
- the quaternionic case $G = Sp(1, n+1)$,

- the octonionic case $G = F_4$,

where the last case is the real form with $K = \text{Spin}(9)$ (this G could be thought of as an analogue of octonionic 3×3 matrices preserving a form of signature $(1, 2)$). In these four cases the flag manifold S is a sphere of dimension $n, 2n + 1, 4n + 3, 8 + 7 = 15$ respectively. Let \mathcal{H}_k be the space of spherical harmonics of degree k , i.e. homogeneous harmonic polynomials of degree k , restricted to S . They form a representation of K , irreducible in the real case. In the other cases we shall explicitly decompose \mathcal{H}_k into irreducible representations of K in order to control the constants in our Sobolev estimates. In all cases the representation π_λ restricted to K is the same as $L^2(S)$ and hence may be identified with the direct sum of all the \mathcal{H}_k , $k \geq 0$.

Just like we can find the spectrum in $L^2(S)$ of Δ , i.e., the usual Laplace–Beltrami operator on S , so can we find the spectrum of the standard Knapp–Stein intertwining operators - see below for more on intertwining operators; this is where we use [3] and [12], which we now recall.

The spherical principal series representations π_λ depend on a single parameter λ , which is in natural duality with $\pi_{-\lambda}$ via the invariant pairing given by integration over K :

$$\langle f, f^* \rangle = \int_K f(k) f^*(k) dk = \int_S f(\xi) f^*(\xi) d\xi$$

where $f(k) = f(kM) = f(\xi)$ is a section of π_λ , resp. f^* a section of $\pi_{-\lambda}$.

A central object in the representation theory of semisimple Lie groups is that of an *intertwining operator*, meaning a G -morphism between two modules (or a morphism for the action of the Lie algebra). There are several standard constructions of such operators, and their analysis is the key to many results about the structure of modules.

We shall be interested in intertwining operators both of integral operator type and differential operator type; the latter occur typically as residues of meromorphic families of intertwining operators of integral operator type. For our purposes the relevant intertwining operators are

$$I_\lambda : V_\lambda \rightarrow V_{-\lambda},$$

where

$$I_\lambda \pi_\lambda(g) = \pi_{-\lambda}(g) I_\lambda$$

for all $g \in G$ (or the analogue for the infinitesimal action of the Lie algebra). Note that in this case the invariant pairing above gives rise to an invariant

Hermitian form on V_λ , namely

$$(f, f) = \langle f, I_\lambda \bar{f} \rangle$$

for $f \in V_\lambda$.

We choose an element $H_0 \in \mathfrak{a}$ with $\alpha(H_0) = 1$, where $\Delta^+ = \{\alpha\}$ (real case), or $\Delta^+ = \{\alpha, 2\alpha\}$ (remaining cases) where we have $m_{2\alpha} = 1, 3, 7$, resp. in the three cases complex, quaternionic, and octonionic. Hence we may identify the parameter $\lambda + \rho$ with a (in general) complex number ν ; this is done by setting $(\lambda + \rho)(H_0) = \nu$. We call the corresponding representation space Y_ν , which may be identified with $L^2(S)$ as a representation of K .

The relevant geometry now is the CR-structure on the sphere S . Let us first in some detail recall the usual CR-structure on S^{2n+1} and the relation to the Heisenberg group H^{2n+1} [9]; this is the complex case in our list above.

We parametrize the Heisenberg group as $\mathbb{C}^n \times \mathbb{R}$ with the group product

$$(z, t)(z', t') = (z + z', t + t' + 2\text{Im } z \cdot \bar{z}'),$$

where $z \cdot \bar{z}' = \sum_{j=1}^n z_j \bar{z}'_j$. We parametrize the Lie algebra in the same way and consider the horizontal subspace H_0 defined by $t = 0$. By left translation L_g , $g \in H^{2n+1}$, the distribution $H_g = dL_g(H_0)$ on H^{2n+1} is defined corresponding to the CR-structure, and the corresponding CR-Laplacian is

$$\Delta_b = \Delta_x + \Delta_y + 4(y \cdot \nabla_x - x \cdot \nabla_y) \frac{\partial}{\partial t} + 4z \cdot \bar{z} \frac{\partial^2}{\partial t^2}$$

in terms of the usual Laplacian and gradient in the variables $x, y \in \mathbb{R}^n$, $z = x + iy$. This operator is hypoelliptic, and it corresponds to the general definition in terms of a contact form; here $\theta = dt + \sum_{j=1}^n (iz_j d\bar{z}_j - i\bar{z}_j dz_j)$, and its Levi form $L_\theta(Z, W) = -id\theta(Z, \bar{W})$. Then

$$\Delta_b = d_b^* d_b, d_b = \pi \circ d : C^\infty(M) \rightarrow H^* = \theta^\perp \subset T^*(M)$$

in general on a CR-manifold; here $M = H^{2n+1}$. Here d is the usual exterior differentiation, and π is the dual to the injection $H \subset TM$. The Levi form defines the inner product on the distribution, which is what is needed to form the adjoint of the horizontal derivative d_b . Again by the inner product, we also have the horizontal gradient ∇_b with values in H_g and $\Delta_b = \nabla_b^* \nabla_b$. Note also that dual is taken with respect to integration over S , which means that we also write

$$\int_S |\nabla_b f(\xi)|^2 d\xi = \int_S (\Delta_b f(\xi)) \overline{f(\xi)} d\xi,$$

where the norm on the left-hand side is taken in the horizontal tangent space.

Now the Cayley transform is defined by $(z_0, z) \rightarrow (w_0, w)$, where

$$w_0 = \frac{z_0 - 1}{z_0 + 1}, \quad w = \frac{2z}{z_0 + 1}$$

and we apply this to $z_0 = it + |z|^2$ resp. z , where (z, t) is an element in H^{2n+1} and $|z|^2 = \sum_{j=1}^n z_j \bar{z}_j$. This gives the stereographic projection (CR-case) of $H^{2n+1} \rightarrow S^{2n+1}$ and also a biholomorphic map from the Siegel domain $\{\operatorname{Re} z_0 > |z|^2\} \rightarrow \{|w_0|^2 + |w|^2 < 1\}$, i.e., the complex unit ball.

On the boundary sphere we get the CR-structure with contact form $\theta = \frac{i}{2} \sum_{j=0}^{n+1} w_j d\bar{w}_j$; the horizontal distribution is given at each tangent space as the maximal complex subspace, and the metric is that induced from the ambient Euclidian space; this will be the normalization to be used. The horizontal gradient is then nothing but the usual gradient of a function, projected orthogonally onto the horizontal space. On $S = S^{2n+1}$ we again have the CR-Laplacian, and, adding a suitable constant, this is an intertwining operator between two principal series representations, namely the CR-Yamabe operator, see [9] and [10]. Below we shall find the spectrum of this operator.

In the quaternionic case, and also the octonionic case, we have in a similar way both a noncompact picture on \bar{N} and a compact picture on $S = K/M$ of a distribution coming from the first summand in $\mathfrak{n} = \mathfrak{n}_\alpha \oplus \mathfrak{n}_{2\alpha}$. On the sphere this is K -invariant and the horizontal gradient ∇_b is again obtained via the Euclidean orthogonal projection. Again, for a suitable constant κ , $\nabla_b^* \nabla_b + \kappa$ is an intertwining operator between two principal series representations.

The first three cases are sometimes called the classical ones; here we recall the key calculations from [12] for the eigenvalues of a G -morphism A_ν (intertwining operator) from Y_ν to its dual, normalized to be 1 on the constant functions:

The real case. On spherical harmonics of degree k the eigenvalue is

$$a_k(\nu) = \prod_{j=1}^k \frac{n-1-\nu+j}{\nu+j-1},$$

and for $\nu = (n-2)/2$ this is proportional to $(k + \frac{n-2}{2})(k + \frac{n}{2})$ which are exactly the expected eigenvalues of the Yamabe operator $\Delta + \frac{n(n-2)}{4}$; in particular we find the well-known spectrum $k(k+n-1)$ for the Laplace operator.

The complex case. The spherical harmonics decompose under K into $\mathcal{H}_k = \sum_{p+q=k} \mathcal{H}^{p,q}$ corresponding to holomorphic type p and anti-holomorphic type q . The eigenvalues are

$$a_{p,q}(\nu) = \prod_{j=1}^p \frac{2n - \nu + 2j}{\nu + 2j - 2} \prod_{l=1}^q \frac{2n - \nu + 2l}{\nu + 2l - 2}.$$

We want to consider the case $\nu = n$ in order to find the eigenvalues of the second-order differential intertwining operator. This gives $(2p+n)(2q+n)$, and subtracting the constant term we obtain the values $4pq + 2(p+q)n = k(k+2n) - j^2$, where $k = p+q$, $j = p-q$, for the eigenvalues of the CR-Laplacian Δ_b . Note that this is consistent with standard calculations, see e.g. [4], where our $\Delta_b = 2\text{Re}\square_b$, $\square_b = \bar{\partial}_b^* \bar{\partial}_b$ in terms of the tangential Cauchy–Riemann complex.

The quaternionic case. The spherical harmonics decompose in this case under K into $\mathcal{H}_k = \sum_q V^{k,q}$ corresponding to certain irreducible representations $V^{p,q}$, $k = p$; the sum is over $p \geq q \geq 0$ and $p - q$ even. We set $r = (p - q)/2$, $s = (p + q)/2$. Then the eigenvalues are

$$a_{p,q}(\nu) = \prod_{j=1}^r \frac{4n + 2 - \nu + 2j}{\nu + 2j - 4} \prod_{l=1}^s \frac{4n + 4 - \nu + 2l}{\nu + 2l - 2},$$

and we are again interested in a particular ν , namely $\nu = 2n + 2$ corresponding to the second-order differential intertwining operator. This gives $(2n + 2r)(2n + 2 + 2s)$; subtracting the constant part we get for the eigenvalues of the CR-Laplacian Δ_b just $k(k + 4n + 2) - j(j + 2)$, where $2s = k + j$, $2r = k - j$, $k = p$, $p - q = j$.

The octonionic case. Here we use the calculations for this group in [3]. We also refer to [11] for the precise relation between spherical harmonics and the K -types occurring in $L^2(S)$, and for more details on the action of K . Again we have the eigenvalues of intertwining operators for the spherical principal series, now found via the method of spectrum generating operators. (In fact, the same eigenvalues are found in [11] by the same method as in [12].) The spectral function (the eigenvalues of the intertwining operator) is in this case

$$Z = a_{k,j}(r) = \frac{\Gamma(j + k + \frac{11}{2} + \frac{r}{2})\Gamma(\frac{11}{2} - \frac{r}{2})\Gamma(k + \frac{5}{2} + \frac{r}{2})\Gamma(\frac{5}{2} - \frac{r}{2})}{\Gamma(j + k + \frac{11}{2} - \frac{r}{2})\Gamma(\frac{11}{2} + \frac{r}{2})\Gamma(k + \frac{5}{2} - \frac{r}{2})\Gamma(\frac{5}{2} + \frac{r}{2})},$$

where $j, k \in \mathbb{N}$ label the K -types; $\mathfrak{k} = \mathfrak{so}(9)$, and we label the representations in the usual way via their highest weight $(\lambda_1, \lambda_2, \lambda_3) = (k + \frac{1}{2}j, \frac{1}{2}j, \frac{1}{2}j)$. Note that $K = \text{Spin}(9)$ and $M = \text{Spin}(7)$ with a nonstandard imbedding. The parameter r here is including the ρ -shift; we have the positive root spaces \mathfrak{g}_α and $\mathfrak{g}_{2\alpha}$ of dimensions 8 and 7 respectively, hence $\rho = 11$ and the above $\nu = 11 - r$. For the second-order differential intertwining operator we have $r = 1$, and the relevant eigenvalues are $4(j + k + 5)(k + 2) - 40$. With $N = j + 2k$ this is equal to $N(N + 14) - j(j + 6)$, where N is exactly the degree of spherical harmonics on S that we decompose under K ; see [11] where the K -types are labeled $V^{N,j}$ with our notation for the parameters, and $N \geq j \geq 0$, $N - j$ even. Summarizing, we have that the eigenvalues of the CR-Laplacian Δ_b in this case are $N(N + 14) - j(j + 6)$, $0 \leq j \leq N$.

2 Logarithmic Sobolev inequalities for rank-one groups

We can now state our main result in this paper.

Theorem 2.1. *Let G be a split rank-one group and $S = G/P = K/M$ the corresponding flag manifold; then with the normalized rotation-invariant measure $d\xi$ on S , we have for any smooth function f on S (and we may extend naturally by taking suitable limits of functions)*

$$\int_S |f(\xi)|^2 \log |f(\xi)| d\xi \leq C \int_S |\nabla_b f(\xi)|^2 d\xi + \|f\|_2^2 \log \|f\|_2 \quad (1)$$

where ∇_b denotes the boundary CR-gradient, and $\|f\|_2$ the usual L^2 -norm. In the four cases the constant is:

- (real case) $C = \frac{1}{n}$, $G = SO_o(1, n + 1)$, $S = S^n$,
- (complex case) $C = \frac{1}{2n}$, $G = SU(1, n + 1)$, $S = S^{2n+1}$,
- (quaternionic case) $C = \frac{1}{4n}$, $G = Sp(1, n + 1)$, $S = S^{4n+3}$,
- (octonionic case) $C = \frac{1}{8}$, $G = F_4$, $S = S^{15}$.

Proof. We shall use the inequality found by Beckner for the sphere S :

$$\int_S |F(\xi)|^2 \log |F(\xi)| \leq \sum_k k \int_S |Y_k(\xi)|^2, \quad (2)$$

for $F = \sum_k Y_k$ decomposed into spherical harmonics; see [1] equation (8). Again we assume F is normalized in L^2 i.e., $\int_S |F|^2 = 1$. This comes from the limit $p = 2$ in the HLS inequality. Now we employ the spectrum of the operator $B = \nabla_b^* \nabla_b = \Delta_b$ (which in the real case is just Δ) in our four cases:

- (real case) on \mathcal{H}_k we have $B = k(k + n - 1)$ and the estimate

$$k \leq \frac{k(k + n - 1)}{n}.$$

- (complex case) on \mathcal{H}_k we have $B = k(k + 2n) - r^2$, $-k \leq r \leq k$ and the estimate

$$k \leq \frac{k(k + 2n) - r^2}{2n}.$$

- (quaternionic case) on \mathcal{H}_k we have $B = k(k + 4n + 2) - j(j + 2)$, $0 \leq j \leq k$ and the estimate

$$k \leq \frac{k(k + 4n + 2) - j(j + 2)}{4n}.$$

- (octonionic case) on \mathcal{H}_k we have $B = k(k + 14) - j(j + 6)$, $0 \leq j \leq k$ and the estimate

$$k \leq \frac{k(k + 14) - j(j + 6)}{8}.$$

These tell us that on each degree k of spherical harmonics we have $k \leq C\Delta_b$ with C as in the theorem, as required. QED

There are many important applications of logarithmic Sobolev inequalities, such as the above; it could be to the Poisson semigroup, to spectral theory, or as in the following example to smoothing properties of the corresponding heat semigroup, in analogy with the contraction properties of the Ornstein–Uhlenbeck semigroup.

Corollary 2.1. *In each of the four cases we have the contraction estimate for the norm of the semigroup ($t \geq 0$), $\exp(-tB) : L^q(S) \rightarrow L^p(S)$,*

$$\|\exp(-tB)\|_{q,p} \leq 1, \text{ for } \exp(-t/C) \leq \sqrt{\frac{q-1}{p-1}}$$

where $B = \Delta_b$ and C has the value as in the theorem.

Proof. This follows from [7] since our Δ_b is a Sobolev generator. QED

3 Inequalities in the noncompact picture

In this section we shall give a new proof of L. Gross' logarithmic Sobolev inequality on \mathbb{R}^n with the Gauss measure, using our main theorem. The idea is to transfer the logarithmic inequality on the sphere to the flat space (Euclidean space in the real case, and a nilpotent group in the CR-case). Consider functions only depending on a fixed number of variables, and let the number of remaining variables tend to infinity. In this way the Gauss measure turns up in the real case and we obtain the classical inequality of L. Gross.

3.1 Stereographic projection in the real case

For this section, it is useful as above to realize the representation π_λ as acting on smooth sections of a line bundle over S , and then to consider the explicit transform to the noncompact picture. In group-theoretic terms we use the orbit of $\overline{N} = \theta(N)$ in G/P to provide coordinates; in this way π_λ is realized in functions on \overline{N} and the logarithmic Sobolev inequality becomes translated into an inequality on N (identified with \overline{N}) equipped with a suitable probability measure and usual CR-structure. Let us first look at the real case (where the group $N = \mathbb{R}^n$, and we are dealing with the usual conformal structure).

The transition from the compact to the noncompact picture is here given by the stereographic projection

$$x = \frac{\xi}{1 + \xi_{n+1}}$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, $\xi_{n+1} \in \mathbb{R}$, $(\xi, \xi_{n+1}) \in S^n$, and the inverse is given by

$$\xi = \frac{2x}{1 + |x|^2}, \quad \xi_{n+1} = \frac{1 - |x|^2}{1 + |x|^2}.$$

This is conformal with conformal factor $1 + \xi_{n+1} = \frac{2}{1 + |x|^2}$. Hence the Euclidean measures on S^n resp. \mathbb{R}^n are related by $d\xi = \left(\frac{2}{1 + |x|^2}\right)^{-n} dx$ and the norm of the gradients will scale in a similar way: Since by definition $|dF|^2 = |\nabla F|^2$, and since the inner product in the cotangent space scales with λ^{-2} when the inner product in the tangent space scales with λ^2 , we obtain the following form in \mathbb{R}^n of the logarithmic Sobolev inequality on the

sphere:

$$c_n \int_{\mathbb{R}^n} |f(x)|^2 \log|f(x)| (1 + |x|^2)^{-n} dx \leq \frac{c_n}{4n} \int_{\mathbb{R}^n} |\nabla f(x)|^2 (1 + |x|^2)^{-n+2} dx$$

for $c_n \int_{\mathbb{R}^n} |f(x)|^2 (1 + |x|^2)^{-n} dx = 1$ and $c_n \int_{\mathbb{R}^n} (1 + |x|^2)^{-n} dx = 1$.

Now we consider the change of variable to x/\sqrt{n} and functions of the form $x \rightarrow f(x/\sqrt{n})$, using $\nabla(f(x/\sqrt{n})) = \frac{1}{\sqrt{n}}(\nabla f)(x/\sqrt{n})$; this gives

$$\begin{aligned} c'_n \int_{\mathbb{R}^n} |f(x)|^2 \log|f(x)| \left(1 + \frac{|x|^2}{n}\right)^{-n} dx \\ \leq \frac{c'_n}{4} \int_{\mathbb{R}^n} |\nabla f(x)|^2 \left(1 + \frac{|x|^2}{n}\right)^{-n+2} dx \end{aligned}$$

for $c'_n \int_{\mathbb{R}^n} |f(x)|^2 \left(1 + \frac{|x|^2}{n}\right)^{-n} dx = 1$ and $c'_n \int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{n}\right)^{-n} dx = 1$.

In order to evaluate the normalization constants needed here and later we record the following.

Lemma 3.1. *For $2N > m$ we have*

$$\int_{\mathbb{R}^m} \left(1 + \frac{|x|^2 + |y|^2}{n}\right)^{-N} dy = n^{m/2} \pi^{m/2} \frac{\Gamma(N - \frac{m}{2})}{\Gamma(N)} \left(1 + \frac{|x|^2}{n}\right)^{-N + \frac{m}{2}}$$

for any $|x|^2 \geq 0$.

As a next step we fix k and let $n = m + k$ be large; assume the function $f(x)$ only depends on the first k variables: $f(x) = f(x_1, x_2, \dots, x_k)$ so that we can perform the integration in the remaining variables first. With the notation $x \in \mathbb{R}^k$, $y \in \mathbb{R}^m$, we calculate

$$\begin{aligned} I &= \int_{\mathbb{R}^m} \left(1 + \frac{|x|^2 + |y|^2}{n}\right)^{-n} dy \\ &= \left(1 + \frac{|x|^2}{n}\right)^{-n} \int_{\mathbb{R}^m} \left(1 + \frac{|y|^2/n}{1 + \frac{|x|^2}{n}}\right)^{-n} dy \end{aligned}$$

where we change variables to $y/\sqrt{(n(1 + \frac{|x|^2}{n}))}$ in order to get

$$I = d_{n,k} \left(1 + \frac{|x|^2}{n}\right)^{(-n-k)/2}$$

with the normalizing constant satisfying

$$d'_{n,k} \int_{\mathbb{R}^k} \left(1 + \frac{|x|^2}{n}\right)^{(-n-k)/2} dx = 1$$

and $d'_{n,k} = c'_n d_{n,k}$. In fact, from the lemma we find

$$\begin{aligned} c'_n &= n^{-n/2} \pi^{-n/2} \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \\ d_{n,k} &= n^{m/2} \pi^{m/2} \frac{\Gamma(\frac{n+k}{2})}{\Gamma(n)} \\ d'_{n,k} &= n^{-k/2} \pi^{-k/2} \frac{\Gamma(\frac{n+k}{2})}{\Gamma(\frac{n}{2})} \end{aligned}$$

and we shall also use below

$$\tilde{d}_{n,k} = n^{m/2} \pi^{m/2} \frac{\Gamma(\frac{n+k}{2} - 2)}{\Gamma(n - 2)}$$

as well as $\tilde{d}'_{n,k} = c'_n \tilde{d}_{n,k}$.

Integrating the y -variable in our inequality we obtain

$$\begin{aligned} & d'_{n,k} \int_{\mathbb{R}^k} |f(x)|^2 \log|f(x)| \left(1 + \frac{|x|^2}{n}\right)^{(-n-k)/2} dx \\ & \leq \frac{\tilde{d}'_{n,k}}{4} \int_{\mathbb{R}^k} |\nabla f(x)|^2 \left(1 + \frac{|x|^2}{n}\right)^{(-n-k)/2+2} dx \end{aligned}$$

for

$$d'_{n,k} \int_{\mathbb{R}^k} |f(x)|^2 \left(1 + \frac{|x|^2}{n}\right)^{(-n-k)/2} dx = 1$$

and (again)

$$d'_{n,k} \int_{\mathbb{R}^k} \left(1 + \frac{|x|^2}{n}\right)^{(-n-k)/2} dx = 1.$$

Now we take the limit $n \rightarrow \infty$, taking into account the asymptotics of $d'_{n,k} \sim (2\pi)^{-k/2}$ and $\tilde{d}'_{n,k}/4 \sim (2\pi)^{-k/2}$ from $\frac{\Gamma(z+a)}{\Gamma(z)} \sim z^a$, $|z| \rightarrow \infty$, and also $(1 + \frac{a}{n})^{-n} \rightarrow e^{-a}$, and we finally obtain

$$\int_{\mathbb{R}^k} |f(x)|^2 \log|f(x)| d\nu(x) \leq \int_{\mathbb{R}^k} |\nabla f(x)|^2 d\nu(x)$$

for the Gauss measure $d\nu(x) = (2\pi)^{-k/2} e^{-|x|^2/2} dx$ and $\int_{\mathbb{R}^k} |f(x)|^2 d\nu(x) = 1$. Hence we have arrived at the logarithmic Sobolev inequality of L. Gross, as in the appendix.

3.2 Cayley transform in the usual CR-case

Here we employ the Cayley transform between the Heisenberg group and the CR-sphere; as in the real case we transform the horizontal gradient and the measure to the Heisenberg group, and get the corresponding form of the logarithmic Sobolev inequality. Recall the explicit coordinate changes

$$w_0 = (z_0 - 1)/(z_0 + 1), \quad w = 2z/(z_0 + 1)$$

where $z_0 = it + |z|^2$ so that the measure, see e.g., [10], on $S = S^{2n+1}$ becomes (up to a constant) $((1 + |z|^2)^2 + t^2)^{-n-1} dz dt$ with dz and dt denoting Lebesgue measures in \mathbb{C}^n resp. \mathbb{R} .

On the Heisenberg group we have the left-invariant CR-holomorphic vector fields

$$Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}$$

corresponding to the distribution; the real and imaginary parts form a basis of the distribution and define the CR-gradient. Explicitly we have the real left-invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$$

for $j = 1, 2, \dots, n$. Then the CR-Laplacian is also $\Delta_b = \sum_1^n (X_j^2 + Y_j^2)$.

We can make the change of variables similar to the real situation as $f(\frac{z}{\sqrt{2n}}, \frac{t}{2n})$; now

$$\nabla_b f\left(\frac{z}{\sqrt{2n}}, \frac{t}{2n}\right) = \frac{1}{\sqrt{2n}} (\nabla_b f)\left(\frac{z}{\sqrt{2n}}, \frac{t}{2n}\right)$$

and furthermore again we have to take into account how the CR-gradient changes by the CR-conformal factor, see [10]. In this way we can write the logarithmic Sobolev inequality in the CR-case on the Heisenberg group as

$$c'_n \int_{\mathbb{C}^n \times \mathbb{R}} |f(z, t)|^2 \log |f(z, t)| d\mu_n(z, t) \leq \frac{c'_n}{4} \int_{\mathbb{C}^n \times \mathbb{R}} |\nabla_b f(z, t)|^2 d\mu_{n-1}(z, t)$$

for $c'_n \int_{\mathbb{C}^n \times \mathbb{R}} |f(z, t)|^2 d\mu_n(z, t) = 1$. The measure is here the (up to the constant c'_n probability) measure

$$d\mu_n(z, t) = \left(\left(1 + \frac{|z|^2}{2n}\right)^2 + \frac{t^2}{4n^2} \right)^{-n-1} dz dt$$

with dz and dt Lebesgue measures as before; here

$$c'_n \int_{\mathbb{C}^n \times \mathbb{R}} d\mu_n(z, t) = 1$$

$$c'_n = (2n)^{-n-1} \pi^{-n-\frac{1}{2}} \frac{\Gamma(2n+1)}{\Gamma(n+\frac{1}{2})}$$

by standard tables, e.g., [5] p. 343 formula 2. This provides a new inequality on the Heisenberg group, and it might be possible to obtain some analogue of the Gaussian logarithmic Sobolev inequality on \mathbb{R}^k as a consequence. We shall refrain from completing this idea, but just limit ourselves to giving a few explicit inequalities that one may immediately deduce.

Now in order to see what happens, we try the trick that we used in the real case, namely that of letting the function only depend on the first k variables. Thus we will write $z = (u, v) \in \mathbb{C}^k \times \mathbb{C}^m$, $n = m + k$ with k fixed and n large, and consider the integral

$$I = \int_{\mathbb{C}^m} \left(\left(1 + \frac{|u|^2 + |v|^2}{2n} \right)^2 + \frac{t^2}{4n^2} \right)^{-N} dv$$

which we evaluate using [5] p. 345, formula 10. The result is

$$I = (2n)^m \left(1 + \frac{|u|^2}{2n} \right)^{-2N+m} \frac{2\pi^m}{\Gamma(m)}$$

$$\times \int_0^\infty \left((1+r^2)^2 + \frac{t^2}{4n^2} \left(1 + \frac{|u|^2}{2n} \right)^{-2} \right)^{-N} r^{2m-1} dr$$

where the last integral for $t = 0$ equals $\frac{1}{2}B(m, 2N - m) = \frac{\Gamma(m)\Gamma(2N-m)}{2\Gamma(2N)}$ in terms of the usual beta function, and in general can be further rewritten as

$$\frac{1}{2}(1+D)^{-N+\frac{m}{2}} \int_0^\infty (x^2 + 2\beta x + 1)^{-N} x^{m-1} dx$$

where $D = C/A^2$, $C = t^2/(2n)^2$, $A = 1 + \frac{|u|^2}{2n}$, $\beta = (1+D)^{-1/2}$.

Now we integrate with respect to the v -variable in the inequality on the Heisenberg group and find the asymptotics for large $n = m + k$ with k fixed. Summarizing, we obtain

$$\frac{1}{\sqrt{n}} \int_{\mathbb{C}^k \times \mathbb{R}} |f|^2 \log |f| d\nu_n \leq \frac{1}{\sqrt{n}} \int_{\mathbb{C}^k \times \mathbb{R}} |\nabla_b f|^2 d\rho_n + 8\sqrt{n} \int_{\mathbb{C}^k \times \mathbb{R}} \left| \frac{\partial f}{\partial t} \right|^2 \widetilde{d\rho_n}$$

for $\frac{1}{\sqrt{n}} \int_{\mathbb{C}^k \times \mathbb{R}} |f|^2 d\nu_n = 1$. Here $\frac{1}{\sqrt{n}} \int_{\mathbb{C}^k \times \mathbb{R}} d\nu_n = 1$ and we have the asymptotic relations

$$d\rho_n(u, t) \sim \widetilde{d\rho_n}(u, t) \sim d\nu_n(u, t) \sim (2\pi)^{-k} e^{-|u|^2/2} \frac{du dt}{\sqrt{2\pi}}$$

for $n \rightarrow \infty$.

Appendix : Hardy–Littlewood–Sobolev inequalities

These are inequalities of the following classical type [8]: Suppose we have two finite sequences of nonnegative real numbers $a_i, b_i, i = 1, \dots, n$, and we consider the sum

$$Q = \sum_{i=1}^n a_i b_i.$$

Then with the same sequences rearranged in decreasing order, $a_1^* \geq a_2^* \cdots \geq a_n^*$, resp. $b_1^* \geq b_2^* \cdots \geq b_n^*$, we have that $Q^* \geq Q$ where now

$$Q^* = \sum_{i=1}^n a_i^* b_i^*.$$

As we see in [8] the same principle of rearrangement may be extended to functions and many other types of expressions; similar ideas are found in other forms of symmetrization, such as e.g., Steiner symmetrization. We shall be interested in quantities of the form

$$Q = \int \int f(x) g(y) h(x - y) dx dy$$

with each integration being over \mathbb{R}^n and dx, dy denote Lebesgue measure.

Then for nonnegative measurable functions f, g, h we consider their equimeasurable symmetric non-increasing rearrangements f^*, g^*, h^* (as in [8]) and have $Q^* \geq Q$ where now

$$Q^* = \int \int f^*(x) g^*(y) h^*(x - y) dx dy$$

[8]. This forms the basis of E. Lieb's [13] deep analysis where he establishes the following sharp Hardy–Littlewood–Sobolev (HLS) inequality (see also [1]):

Proposition 3.1. *Let $S = S^n$ be the n -dimensional sphere with the normalized usual rotation-invariant measure, $0 < \lambda < n$, $0 < p = \frac{2n}{2n-\lambda} < 2$; then we have the estimate for the L^p -norm $\|F\|_p = (\int_S |F|^p)^{1/p}$*

$$\sum_{k=0}^{\infty} \gamma_k \int_S |Y_k|^2 \leq \|F\|_p^2,$$

where $F = \sum_{k=0}^{\infty} Y_k$ is the decomposition of the measurable function F into spherical harmonics $Y_k \in \mathcal{H}_k$ of degree k ,

$$\gamma_k = \frac{\Gamma(\frac{n}{p})\Gamma(\frac{n}{p'} + k)}{\Gamma(\frac{n}{p'})\Gamma(\frac{n}{p} + k)}$$

and p, p' are dual exponents: $\frac{1}{p} + \frac{1}{p'} = 1$.

This is equivalent to giving the best constant

$$K_p = \pi^{n/p'} \frac{\Gamma(\frac{n}{p} - \frac{n}{2})}{\Gamma(\frac{n}{p})} \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right)^{(p-2)/p}$$

in the estimate

$$\int \int f(x) |x - y|^{-\lambda} g(y) dx dy \leq K_p \|f\|_p \|g\|_p$$

for nonnegative functions and their L^p norms, where the measures dx, dy are Lebesgue measure and the integrals over \mathbb{R}^n .

Note that the integral $If(y) = \int f(x) |x - y|^{-\lambda} dx$ defines an intertwining operator between two principal series representations of $G = SO(1, n + 1)$, namely from a representation to its natural dual. It is a very important part of the theory that one can find the eigenvalues on the sphere, that is, in the compact picture of the principal series representations of this family of intertwining operators. Note that when we make a change of variables and transform I to the sphere and normalize it so that $I1 = 1$, the sharp HLS inequality states that $\|If\|_{p'} \leq \|f\|_p$. It is an appealing conjecture, that such a contraction property holds more generally, i.e., for other groups and their continuations of principal series to suitable real parameters. The representations here belong to the complementary series and they are unitary through the invariant Hermitian form coming from the intertwining operator. On the other hand, the $L^p(S)$ Banach norm is invariant in the original space, and the $L^{p'}(S)$ Banach norm is invariant in the target space.

So another way to think of the sharp HLS inequality is the following for the invariant unitary norm $\|f\|$ (say for real functions):

$$\|f\|^2 = \langle If, f \rangle \leq \|If\|_{p'} \|f\|_p \leq \|f\|_p^2.$$

The conjecture would be, that this remains true in the CR-case, where the complex case is already interesting.

Now as demonstrated in [1] it is very interesting to study the parameter endpoints $p = 1, 2$ in HLS, where one may consider the derivatives in the parameter. One result that follows at $p = 2$ is the celebrated logarithmic Sobolev inequality below [6] for the Gauss measure $d\nu = (2\pi)^{-n/2} e^{-|x|^2/2} dx$ on \mathbb{R}^n . There are several different proofs of this result, and in this paper we have given a new way of deriving it from the real case in our main Theorem. At $p = 1$ one obtains [1] an exponential-class inequality of Moser-Trudinger type. It is a highly interesting problem to find the right analogues of exponential-class inequalities in the framework of the CR-geometries considered in this paper.

Proposition 3.2. *For a smooth function f on \mathbb{R}^n (or suitable limit functions) we have the estimate*

$$\int |f|^2 \log |f| d\nu \leq \int |\nabla f|^2 d\nu$$

for $\int |f|^2 d\nu = 1$.

More generally, if we have a probability space (Ω, μ) and a self-adjoint linear operator B on $L^2(\mu)$ with $B \geq 0$ satisfying

$$\int_{\Omega} |f|^2 \log |f| d\mu \leq (Bf, f)$$

for all f in the domain of B with $\|f\|_2 = 1$, then we call this a logarithmic Sobolev inequality with *Sobolev generator* B .

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